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Statistical interpretation of the turbulent dissipation rate in wall-bounded flows

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Statistical analysis was performed for interpreting the dissipation correlations in turbulent wall-bounded flows. The fundamental issues related to the formulation of the closure assumptions are discussed. Using the two-point correlation technique, a distinction is made between the homogeneous and inhomogeneous parts of the dissipation tensor. It is shown that the inhomogeneous part contributes half of the dissipation rate at the wall and vanishes remote from the wall region. The structure of an analytically derived equation was analysed utilizing the results of direct numerical simulations of turbulent channel flow at low Reynolds number.

1. Introduction

In 1945, Chou (1945) outlined a general concept for the closure of the Reynolds equations for turbulent flow. He identified three different types of correlations involved in the equations for the moments: higher-order velocity correlations, velocity/pressure gradient correlations and dissipation correlations. Inspired by the work of von Kármán and Lin, Chou used the two-point correlation technique together with the assumption of local homogeneity to develop a systematic and consistent statistical procedure for the closure of velocity/pressure gradient and dissipation correlations. The successive substitution method was proposed for the treatment of the higher-order velocity correlations by solving the dynamic equations for the moments throughout the iteration.

Rotta (1951) followed the work of Chou and deduced explicit values for the empirical constants by referring to the experimental data. He used these data to predict the entire non-vanishing components of the Reynolds stress tensor in a two-dimensional channel flow with reasonable success. It must be stressed, however, that Rotta modified Chou's original theory and used simplified forms for the velocity/pressure gradient and dissipation correlations.

The closure for the equations that describe second-order moments of turbulent velocity fluctuations was also considered by Davydov (1961). He incorporated the isotropy assumptions for the dissipation rate correlations and formulated approximate relationships for the pressure–strain terms and correlations encountered in the equations for the third-order moments. In addition to the improvements mentioned above, Davydov defined a transport equation for the turbulent dissipation rate. For the two dominant terms of this equation, he provided a closure based on the laws of decay of homogeneous isotropic turbulence. The closure procedure proposed by

Davydov requires the integration of 23 differential equations and determination of four empirical constants.

In the late 1960s and the beginning of the 1970s, a group of physicists at the Los Alamos Research Laboratories directed by Harlow (e.g. Harlow & Welch 1965; Harlow & Nakayama 1968; Harlow & Hirt 1969; Hirt 1969; Daly & Harlow 1970) developed several efficient numerical techniques for fluid flow applications. They implemented the modified turbulence models of Rotta and Davydov in numerical computation schemes and obtained predictions of simple flow configurations. Parametric studies permitted optimization of the empirical constants by matching computational results to the available experimental data.

Further refining and testing of turbulence closure schemes were carried out by Hanjalić & Launder (1972, 1976). They extensively studied thin shear flows utilizing Davydov's formulation for the turbulence dissipation rate and modified Rotta's suggestion for the pressure–strain correlations. A simplified set of equations was solved for the prediction of various wall flows using the equations for the shear stress, turbulence kinetic energy and dissipation rate. In a later publication, they also included low Reynolds number effects, which are of significant importance for the treatment of the near-wall region.

The first numerical evaluations employing a complete treatment for all of the non-vanishing components of the correlation tensor were realized by Launder, Reece & Rodi (1975) exactly 30 years after the appearance of the original publication of Chou. The turbulence closure was a mixture of Chou's original ideas and the later work of Rotta, Davydov, Harlow, and Hanjalić. Satisfactory results were obtained by comparing predictions against available experimental data in some specific cases of homogeneous, free shear flows and turbulent wall boundary layers.

A significant intellectual contribution to the formulation of the turbulence closure was made by Lumley (1978). He formulated a novel approach for handling the dynamic equations for the moments based on the analogy between the behaviour of turbulent flows and viscoelastic fluids. Lumley (1992) explicitly underlined the need for an accurate description of the processes related to the dissipation of kinetic energy of turbulence as a most crucial item for reliable flow predictions.

With the advances in the development of direct simulation techniques for turbulence investigations, it is now possible to test various mathematical theories of turbulence directly against the simulation databases. These databases contain complete three-dimensional random flow fields from which it is possible to extract any information that is required. Thus, we can avoid putting effort into handling large systems of partial differential equations and concentrate attention on checking physical ideas and fundamental assumptions that are currently used in the development of turbulence closure.

The objective of this paper is to analyse in great detail the structure of the equations that govern the turbulent dissipation rate. The two-point correlation technique first introduced by Chou (1945) and subsequently improved by Kolovandin & Vatutin (1969, 1972) forms the bases of our theoretical analysis. The result of this analysis is an equation for the homogeneous part of the dissipation which relates to the total dissipation by a simple algebraic relationship. The derived equation was analysed using the results of direct numerical simulations of turbulent channel flow. By considering the equation for the dissipation rate, additional arguments are supplemented that demonstrate the obvious advantages of the derived equation for application in wall-bounded flows.

2. Basic equations for the moments

Starting from the Navier–Stokes and the continuity equations and introducing the conventional method of separating the instantaneous velocity and pressure into mean and fluctuating components, one obtains the equations for the turbulent fluctuations (see Hinze 1975):

$$\frac{\partial u_i}{\partial t} + U_k \frac{\partial u_i}{\partial x_k} + u_k \frac{\partial U_i}{\partial x_k} + \frac{\partial u_i u_k}{\partial x_k} - \frac{\partial \overline{u_i u_k}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta_x u_i, \tag{2.1}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2.2}$$

In the above equations, the summation convention is applied over all double indices, and Δ_x corresponds to the Laplace operator ($\Delta_x = \partial^2/\partial x_l \partial x_l$) with respect to the variable x.

By systematic manipulation of (2.1) and (2.2), it is possible to obtain equations for the moments of arbitrary order (see, for example, Hinze 1975). The equations defining the second-order moments are

$$\frac{\partial \overline{u_i u_j}}{\partial t} + U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} + \overline{u_i u_k}\frac{\partial U_j}{\partial x_k}$$
$$+ \underbrace{\frac{\partial \overline{u_i u_j u_k}}{\partial x_k}}_{(I)} + \frac{1}{\rho}\underbrace{[\overline{u_j \frac{\partial p}{\partial x_i}} + \overline{u_i \frac{\partial p}{\partial x_j}}]}_{(II)} + 2\nu \underbrace{\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}}_{(III)} - \nu \Delta_x \overline{u_i u_j} = 0. \tag{2.3}$$

In the equations for the second-order moments, one can identify three different types of unknown correlations: higher-order velocity correlations (I); velocity/pressure gradient correlations (II); dissipation correlations (III).

These correlations must be expressed in terms of known quantities in order to close the resultant system of equations for the moments.

In this paper, we consider the terms representing the dissipation of turbulence:

$$\epsilon_{ij} = \nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}, \tag{2.4}$$

that appear in (2.3). The procedure for treating the dissipation correlations is based on the application of the two-point correlation technique that was originally developed by Chou (1945) and subsequently refined by Kolovandin & Vatutin (1972).

3. Application of the two-point correlation technique for interpreting dissipation correlations

In order to separate the effects of local character from the large-scale fluid motions, we first define a new coordinate system relative to two arbitrary points A and B as shown in figure 1 :

$$\xi_k = (x_k)_B - (x_k)_A, \tag{3.1}$$

$$(x_k)_{AB} = \tfrac{1}{2}[(x_k)_A + (x_k)_B]. \tag{3.2}$$

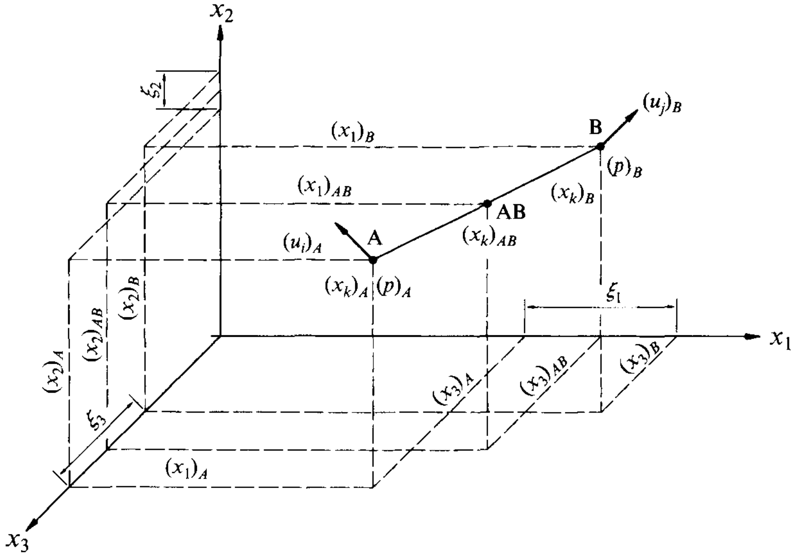

FIGURE 1. Coordinate system used to define the two-point correlation functions.

The partial differential operators at points A and B as functions of $(x_k)_{AB}$ and ξ_k are given as follows (see Hinze 1975):

$$\left(\frac{\partial}{\partial x_k}\right)_A = \left(\frac{\partial}{\partial x_k}\right)_{AB} \frac{\partial (x_k)_{AB}}{\partial (x_k)_A} + \frac{\partial}{\partial \xi_k}\frac{\partial \xi_k}{\partial (x_k)_A} = \frac{1}{2}\left(\frac{\partial}{\partial x_k}\right)_{AB} - \frac{\partial}{\partial \xi_k}, \tag{3.3}$$

$$\left(\frac{\partial}{\partial x_k}\right)_B = \left(\frac{\partial}{\partial x_k}\right)_{AB} \frac{\partial (x_k)_{AB}}{\partial (x_k)_B} + \frac{\partial}{\partial \xi_k}\frac{\partial \xi_k}{\partial (x_k)_B} = \frac{1}{2}\left(\frac{\partial}{\partial x_k}\right)_{AB} + \frac{\partial}{\partial \xi_k}. \tag{3.4}$$

From the last two equations, one obtains

$$\left(\frac{\partial}{\partial x_k}\right)_A \left(\frac{\partial}{\partial x_k}\right)_B = \frac{1}{4}\left(\frac{\partial^2}{\partial x_k \partial x_k}\right)_{AB} - \frac{\partial^2}{\partial \xi_k \partial \xi_k}. \tag{3.5}$$

If we now apply the operator (3.5) to the product of velocity fluctuations at two points $(u_i)_A(u_j)_B$, we obtain

$$\left(\frac{\partial}{\partial x_k}\right)_A \left(\frac{\partial}{\partial x_k}\right)_B (u_i)_A(u_j)_B = \frac{1}{4}\left(\frac{\partial^2}{\partial x_k \partial x_k}\right)_{AB} (u_i)_A(u_j)_B - \frac{\partial^2}{\partial \xi_k \partial \xi_k}(u_i)_A(u_j)_B. \tag{3.6}$$

Since $(u_i)_A$ can be treated as constant with respect to a derivative at point B and $(u_j)_B$ constant with respect to a derivative at point A, (3.6) can be transformed, after averaging has been performed, to the following form:

$$\overline{\left(\frac{\partial u_i}{\partial x_k}\right)_A \left(\frac{\partial u_j}{\partial x_k}\right)_B} = \frac{1}{4}\left(\frac{\partial^2}{\partial x_k \partial x_k}\right)_{AB} \overline{(u_i)_A(u_j)_B} - \frac{\partial^2}{\partial \xi_k \partial \xi_k}\overline{(u_i)_A(u_j)_B}. \tag{3.7}$$

Multiplying (3.7) by ν and taking the limit when A $\rightarrow$ B yields

$$\epsilon_{ij} = \nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} = \underbrace{\tfrac{1}{4}\nu \Delta_x \overline{u_i u_j}}_{\text{inhomogeneous}} \; \underbrace{- \nu(\Delta_\xi \overline{u_i u_j'})_0}_{\text{homogeneous}}, \tag{3.8}$$

where the prime indicates a value of the function at the point B and the subscript $_0$ represents the zero separation $\xi = 0$. Equation (3.8) is due to Kolovandin & Vatutin (1972).

The derived form of ϵ_{ij} possesses the following important properties:

(i) According to (3.8), the tensor ϵ_{ij} is composed of an inhomogeneous part $\frac{1}{4}\nu\Delta_x\overline{u_iu_j}$ and a homogeneous part $-\nu(\Delta_\xi\overline{u_iu_j'})_0$.

(ii) Since we derived (3.8) only from kinematic considerations and without reference to the laws of conservation, it is not appropriate to relate the inhomogeneous part of ϵ_{ij} to the viscous diffusion.

(iii) The tensor ϵ_{ij} is symmetrical and from (3.8) it follows that

$$(\Delta_\xi\overline{u_iu_j'})_0 = (\Delta_\xi\overline{u_ju_i'})_0. \tag{3.9}$$

The above result is also a peculiarity of the two-point velocity correlation of second rank in homogeneous turbulence.

(iv) The homogeneous part of ϵ_{ij} depends on viscosity and the curvature of the two-point velocity correlation of second rank near the origin. Hence this part of ϵ_{ij} can be expressed in terms of the viscosity, second-order statistics of the velocity field and the tensor of the turbulence micro-scale (see Kolovandin & Vatutin 1972; Jovanović, Ye & Durst 1992).

(v) By expanding the instantaneous velocities in a Taylor series near the wall, it is possible to show that the inhomogeneous part of ϵ_{ij} is especially important in wall-bounded flows. This part *contributes to half of the dissipation rate* ϵ:

$$\epsilon = \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}}, \tag{3.10}$$

at the wall and vanishes remote from the wall region. It also increases the anisotropy between the components of the dissipation tensor. Hence the inhomogeneous part of (3.8) plays an important role in the partition of ϵ_{ij} into its components at the wall. We shall show later that the data obtained from direct numerical simulations of wall-bounded flows confirm these results (see figure 2). In the context discussed above we may note that the total average turbulent dissipation rate $\bar{\epsilon}$ is defined by

$$\bar{\epsilon} = \nu\overline{\frac{\partial u_i}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)} = \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} + \nu\frac{\partial^2\overline{u_iu_k}}{\partial x_i\partial x_k}. \tag{3.11}$$

(vi) Using (3.8), we can also explain the difficulties in direct treatment of the equation for the dissipation of turbulence kinetic energy. Starting from (3.8), we can write

$$\frac{\mathrm{D}\epsilon_{ij}}{\mathrm{D}t} = \tfrac{1}{4}\nu\Delta_x\tfrac{\mathrm{D}}{\mathrm{D}t}\overline{u_iu_j} - \nu\tfrac{\mathrm{D}}{\mathrm{D}t}(\Delta_\xi\overline{u_iu_j'})_0. \tag{3.12}$$

The substantial derivatives of $\overline{u_iu_j}$ and $(\Delta_\xi\overline{u_iu_j'})_0$ in (3.12) can be replaced by (2.3) and (4.3) to show that *we must keep the terms that are fourth-order derivatives in* $\overline{u_iu_j}$ *to retain the inhomogeneous part of* ϵ_{ij}. This is not practical from the viewpoint of theory and is also inconvenient for the numerical analysis. Therefore, operation upon the equation for dissipation ϵ_{ij} is extremely difficult. In the sections that follow, we shall elaborate further on this important point.

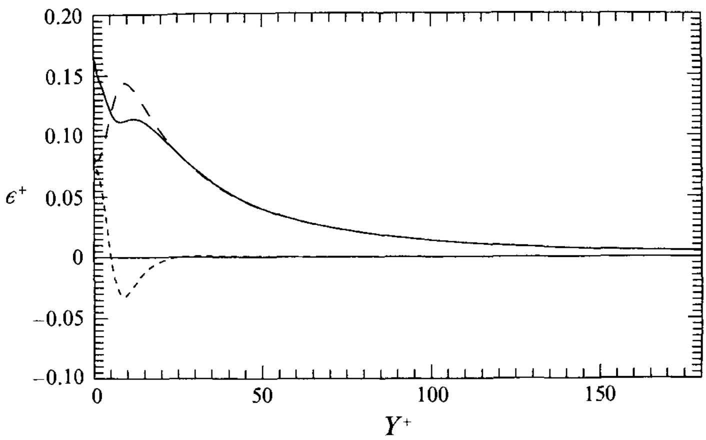

FIGURE 2. Distributions of ϵ, homogeneous and inhomogeneous contributions: ———, ϵ^+; — — — —, homogeneous part of (6.1); - - - - - -, inhomogeneous part of (6.1).

4. Dynamic equations for the two-point velocity correlation

Since the inhomogeneous part of ϵ_{ij} can be explicitly related to the statistics of the velocity field, we need to consider only the homogeneous part of the turbulence dissipation. To achieve this task, we start from the equation for the instantaneous fluctuations, (2.1), and, referencing it to the points A and B, we obtain the following equation, after some manipulations (see Hinze 1975):

$$\begin{aligned}
&\frac{\partial}{\partial t}(u_i)_A(u_j)_B + (u_j)_B(u_k)_A\left(\frac{\partial U_i}{\partial x_k}\right)_A + (u_i)_A(u_k)_B\left(\frac{\partial U_j}{\partial x_k}\right)_B + (U_k)_A\left(\frac{\partial}{\partial x_k}\right)_A \\
&\times (u_i)_A(u_j)_B + (U_k)_B\left(\frac{\partial}{\partial x_k}\right)_A (u_i)_A(u_j)_B = -\left(\frac{\partial}{\partial x_k}\right)_A (u_j)_B(u_i)_A(u_k)_A - \left(\frac{\partial}{\partial x_k}\right)_B \\
&\times (u_i)_A(u_j)_B(u_k)_B + \left(\frac{\partial}{\partial x_k}\right)_A (u_j)_B\overline{(u_iu_k)_A} + \left(\frac{\partial}{\partial x_k}\right)_B (u_i)_A\overline{(u_ju_k)_B} - \frac{1}{\rho}\left[\left(\frac{\partial}{\partial x_i}\right)_A\right. \\
&\left.\times (p)_A(u_j)_B + \left(\frac{\partial}{\partial x_j}\right)_B (p)_B(u_i)_A\right] + \nu\left[\left(\frac{\partial^2}{\partial x_l\partial x_l}\right)_A + \left(\frac{\partial^2}{\partial x_l\partial x_l}\right)_B\right](u_i)_A(u_j)_B. \qquad (4.1)
\end{aligned}$$

Utilizing (3.3), (3.4) and (3.5) derived for the relative coordinate system given in figure 1, and by applying an averaging procedure with respect to time, (4.1) reads

$$\begin{aligned}
&\frac{\partial}{\partial t}\overline{(u_i)_A(u_j)_B} + \overline{(u_j)_B(u_k)_A}\left(\frac{\partial U_i}{\partial x_k}\right)_A + \overline{(u_i)_A(u_k)_B}\left(\frac{\partial U_j}{\partial x_k}\right)_B + \tfrac{1}{2}[(U_k)_A \\
&+ (U_k)_B]\left(\frac{\partial}{\partial x_k}\right)_{AB}\overline{(u_i)_A(u_j)_B} + [(U_k)_B - (U_k)_A]\frac{\partial}{\partial \xi_k}\overline{(u_i)_A(u_j)_B} + \frac{1}{2}\left(\frac{\partial}{\partial x_k}\right)_{AB} \\
&\times [\overline{(u_iu_k)_A(u_j)_B} + \overline{(u_i)_A(u_ju_k)_B}] + \frac{\partial}{\partial \xi_k}[\overline{(u_i)_A(u_ju_k)_B} - \overline{(u_iu_k)_A(u_j)_B}] \\
&+ \frac{1}{2\rho}\left[\left(\frac{\partial}{\partial x_i}\right)_{AB}\overline{(p)_A(u_j)_B} + \left(\frac{\partial}{\partial x_j}\right)_{AB}\overline{(u_i)_A(p)_B}\right] - \frac{1}{\rho}\left[\frac{\partial}{\partial \xi_i}\overline{(p)_A(u_j)_B}\right. \\
&\left. - \frac{\partial}{\partial \xi_j}\overline{(u_i)_A(p)_B}\right] - \frac{\nu}{2}\Delta_x\overline{(u_i)_A(u_j)_B} - 2\nu\Delta_\xi\overline{(u_i)_A(u_j)_B} = 0. \qquad (4.2)
\end{aligned}$$

By decomposing the velocity/pressure gradient correlations in a similar way as described in § 3 and setting $\xi = 0$, it is straightforward to show that (4.2) coincides with (2.3).

To permit the explicit evaluation of the second term in (3.8), we apply the operator $-\nu\Delta_\xi$ to (4.2) and take the limit $A \to B$ to obtain

$$
\begin{aligned}
&-\nu\frac{\partial}{\partial t}(\Delta_\xi\overline{u_iu_j'})_0 - \nu U_k\frac{\partial}{\partial x_k}(\Delta_\xi\overline{u_iu_j'})_0 - \nu(\Delta_\xi\overline{u_ku_j'})_0\frac{\partial U_i}{\partial x_k} - \nu(\Delta_\xi\overline{u_iu_k'})_0\frac{\partial U_j}{\partial x_k}\\
&-\frac{\nu}{4}\left[\overline{u_iu_k}\Delta_x\frac{\partial U_j}{\partial x_k} + \overline{u_ku_j}\Delta_x\frac{\partial U_i}{\partial x_k} + (\Delta_x U_k)\frac{\partial}{\partial x_k}\overline{u_iu_j}\right] - \nu\left(\frac{\partial}{\partial \xi_l}\overline{u_iu_k'}\right)_0\frac{\partial^2 U_j}{\partial x_l\partial x_k}\\
&+\nu\left(\frac{\partial}{\partial \xi_l}\overline{u_ku_j'}\right)_0\frac{\partial^2 U_i}{\partial x_l\partial x_k} - 2\nu\left(\frac{\partial^2}{\partial \xi_l\partial \xi_k}\overline{u_iu_j'}\right)_0\frac{\partial U_k}{\partial x_l} - \frac{\nu}{2}\frac{\partial}{\partial x_k}[(\Delta_\xi\overline{u_iu_ku_j'})_0\\
&+(\Delta_\xi\overline{u_iu_j'u_k'})_0] - \nu\left[\Delta_\xi\frac{\partial}{\partial \xi_k}\left(\overline{u_iu_j'u_k'} - \overline{u_iu_ku_j'}\right)\right]_0 - \frac{\nu}{2\rho}\left[\frac{\partial}{\partial x_i}(\Delta_\xi\overline{pu_j'})_0 + \frac{\partial}{\partial x_j}(\Delta_\xi\overline{u_ip'})_0\right]\\
&+\frac{\nu}{\rho}\left[\Delta_\xi\left(\frac{\partial}{\partial \xi_i}\overline{pu_j'} - \frac{\partial}{\partial \xi_j}\overline{u_ip'}\right)\right]_0 + \frac{\nu^2}{2}\Delta_x(\Delta_\xi\overline{u_iu_j'})_0 + 2\nu^2(\Delta_\xi\Delta_\xi\overline{u_iu_j'})_0 = 0. \qquad (4.3)
\end{aligned}
$$

Using the two-point correlation technique, kinematic constraints and the continuity equation, the components of the dissipation tensor, (3.8), can be analytically interpreted in terms of its trace ϵ_{ss} and the second-order velocity correlation $\overline{u_iu_j}$ (see Chou 1945; Kolovandin & Vatutin 1972, Jovanović *et al.* 1992). Therefore, it is of interest to consider only the contracted form of (4.3), which reads as follows:

$$
\begin{aligned}
&-\nu\frac{\partial}{\partial t}(\Delta_\xi\overline{u_su_s'})_0 - \nu U_k\frac{\partial}{\partial x_k}(\Delta_\xi\overline{u_su_s'})_0 - \nu[(\Delta_\xi\overline{u_ku_s'})_0 + (\Delta_\xi\overline{u_su_k'})_0]\frac{\partial U_s}{\partial x_k}\\
&-\frac{\nu}{4}\left[2\overline{u_su_k}\Delta_x\frac{\partial U_s}{\partial x_k} + (\Delta_x U_k)\frac{\partial}{\partial x_k}\overline{u_su_s}\right] - \nu\left[\left(\frac{\partial}{\partial \xi_l}\overline{u_su_k'}\right)_0 - \left(\frac{\partial}{\partial \xi_l}\overline{u_ku_s'}\right)_0\right]\frac{\partial^2 U_s}{\partial x_l\partial x_k}\\
&-2\nu\left(\frac{\partial^2}{\partial \xi_l\partial \xi_k}\overline{u_su_s'}\right)_0\frac{\partial U_k}{\partial x_l} - \frac{\nu}{2}\frac{\partial}{\partial x_k}[(\Delta_\xi\overline{u_su_ku_s'})_0 + (\Delta_\xi\overline{u_su_s'u_k'})_0]\\
&-\nu\left[\Delta_\xi\frac{\partial}{\partial \xi_k}\left(\overline{u_su_s'u_k'} - \overline{u_su_ku_s'}\right)\right]_0 - \frac{\nu}{2\rho}\frac{\partial}{\partial x_s}[(\Delta_\xi\overline{pu_s'})_0 + (\Delta_\xi\overline{u_sp'})_0]\\
&+\frac{\nu}{\rho}\left[\Delta_\xi\frac{\partial}{\partial \xi_s}\left(\overline{pu_s'} - \overline{u_sp'}\right)\right]_0 + \frac{\nu^2}{2}\Delta_x(\Delta_\xi\overline{u_su_s'})_0 + 2\nu^2(\Delta_\xi\Delta_\xi\overline{u_su_s'})_0 = 0. \qquad (4.4)
\end{aligned}
$$

In the Appendix, we have provided the derivation of (4.4) starting from the equation for the turbulent dissipation rate.

5. Physical simplification of the dynamic equation for the homogeneous part of the turbulent dissipation rate

Equation (4.4) for the homogeneous part of the turbulent dissipation rate is composed of the derivatives of two-point correlation functions. Since this equation involves velocity and pressure/velocity correlations that are of diverse nature, further simplifications are desirable.

The formalism usually adopted in the treatment (4.4) is to perform an order of magnitude analysis of each individual term and to disregard terms that are negligible. However, such an estimation does not recognize the statistical nature of turbulence fluctuations. In 1945, Chou proposed an analytical technique for the treatment of the equations that define the dissipation correlations. He introduced the assumption of local homogeneity for the small-scale structure of turbulence which permitted him to make radical simplifications to (4.4). The concept of local homogeneity is statistical and utilizes as an approximation the relationships for the derivatives of the two-point correlation functions for zero separation ($\xi = 0$) that are valid in homogeneous turbulence. We shall follow this approach in defining the approximate equation that describes the homogeneous part of the turbulent dissipation rate.

To approximate the terms in (4.4) that involve differentiation with respect to the variable ξ following the suggestion of Chou (1945), we assume that the small-scale structure of turbulence between two closely separated points A and B is locally homogeneous. Exploring this assumption, we can utilize the properties of homogeneous turbulence for two-point correlations (see Hinze 1975):

$$\overline{u_s u_s' u_k'} = -\overline{u_s u_k u_s'}, \tag{5.1a}$$

$$\overline{p u_s'} = -\overline{u_s p'}, \tag{5.1b}$$

$$\overline{u_s u_k'} = \overline{u_k u_s'}, \tag{5.1c}$$

differentiate (5.1*a*,*b*,*c*) with respect to ξ, and set A $\rightarrow$ B to show the following:

$$\left(\frac{\partial}{\partial \xi_l}\overline{u_s u_k'}\right)_0 - \left(\frac{\partial}{\partial \xi_l}\overline{u_k u_s'}\right)_0 \simeq 0, \tag{5.2a}$$

$$(\Delta_\xi \overline{u_s u_k u_s'})_0 + (\Delta_\xi \overline{u_s u_s' u_k'})_0 \simeq 0, \tag{5.2b}$$

$$\left(\Delta_\xi \frac{\partial}{\partial \xi_k}\overline{u_s u_s' u_k'}\right)_0 \simeq -\left(\Delta_\xi \frac{\partial}{\partial \xi_k}\overline{u_s u_k u_s'}\right)_0, \tag{5.2c}$$

$$(\Delta_\xi \overline{p u_s'})_0 + (\Delta_\xi \overline{u_s p'})_0 \simeq 0. \tag{5.2d}$$

We shall now complement the derivations given above for the terms of equation (4.4) that involve derivatives of two-point pressure/velocity correlations. Starting from (3.3) and (3.4), we can write

$$\left(\frac{\partial}{\partial x_s}\right)_B - \left(\frac{\partial}{\partial x_s}\right)_A = 2\frac{\partial}{\partial \xi_s}. \tag{5.3}$$

Applying the differential operator (5.3) to the correlations $\overline{(p)_A(u_s)_B}$ and $\overline{(u_s)_A(p)_B}$, and utilizing the conditions of invariance (5.1*b*) together with the continuity equation (2.2) results in

$$\frac{\partial}{\partial \xi_s}[\overline{(p)_A(u_s)_B} - \overline{(u_s)_A(p)_B}] \simeq 0. \tag{5.4}$$

If we apply the Laplace operator Δ_ξ to (5.4) and take the limit $A \rightarrow B$, the result is as follows:

$$\left[\Delta_\xi \frac{\partial}{\partial \xi_s}\left(\overline{p u_s'} - \overline{u_s p'}\right)\right]_0 \simeq 0. \tag{5.5}$$

Using (3.9) and the approximate relationships (5.2*a*,*b*,*c*,*d*) and (5.5), we can simplify (4.4) to

$$
\begin{aligned}
&- \nu \frac{\partial}{\partial t}(\Delta_\xi \overline{u_s u_s'})_0 - \nu U_k \frac{\partial}{\partial x_k}(\Delta_\xi \overline{u_s u_s'})_0 - 2\nu(\Delta_\xi \overline{u_k u_s'})_0 \frac{\partial U_s}{\partial x_k} \\
&- \frac{\nu}{4}\left[2\overline{u_s u_k}\Delta_x \frac{\partial U_s}{\partial x_k} + (\Delta_x U_k)\frac{\partial}{\partial x_k}\overline{u_s u_s}\right] - 2\nu\left(\frac{\partial^2}{\partial \xi_l \partial \xi_k}\overline{u_s u_s'}\right)_0 \frac{\partial U_k}{\partial x_l} \\
&+ 2\nu\left(\Delta_\xi \frac{\partial}{\partial \xi_k}\overline{u_s u_k u_s'}\right)_0 + \frac{\nu^2}{2}\Delta_x(\Delta_\xi \overline{u_s u_s'})_0 + 2\nu^2(\Delta_\xi \Delta_\xi \overline{u_s u_s'})_0 \simeq 0. \qquad (5.6)
\end{aligned}
$$

This simplified equation is identical with that obtained by Kolovandin & Vatutin (1972) and very similar to the equation for the vorticity decay analysed by Chou (1945). In his initial work, Chou ignored the terms

$$
-\frac{\nu}{4}\left[2\overline{u_s u_k}\Delta_x \frac{\partial U_s}{\partial x_k} + (\Delta_x U_k)\frac{\partial}{\partial x_k}\overline{u_s u_s}\right] \quad \text{and} \quad \frac{\nu^2}{2}\Delta_x(\Delta_\xi \overline{u_s u_s'})_0,
$$

that appear in (5.6) on the grounds that they are negligible at high Reynolds numbers and away from the near-wall region.

6. Validation of the derived equations using the results of direct numerical simulations

In this section, we shall utilize the data of Mansour, Kim & Moin (1987) together with the results of Gilbert & Kleiser (1991) with the following objectives: (i) to decompose the inhomogeneous and homogeneous contributions to the turbulent dissipation rate; (ii) to validate the assumption of local homogeneity across the entire flow field and, in particular, near the wall; (iii) to analyse the structure of the approximate equation for the homogeneous part of ϵ and the contributions of each individual term. The test data were obtained from direct numerical simulation of a fully developed turbulent channel flow at low Reynolds numbers.

6.1. *Decomposition of the turbulent dissipation rate*

From the analysis presented in § 3, the turbulent dissipation rate can be decomposed into inhomogeneous and homogeneous parts:

$$
\epsilon = \underbrace{\tfrac{1}{4}\nu\Delta_x q^2}_{\text{inhomogeneous}} \ \underbrace{-\nu(\Delta_\xi \overline{u_s u_s'})_0}_{\text{homogeneous}}. \qquad (6.1)
$$

In figure 2, we have plotted both of these contributions and their sum using the data of Mansour *et al.* (1987). The data are presented versus normalized distance (y^+) from the wall. Normalization was performed with respect to the inner variables

$$
\epsilon^+ = \frac{\epsilon\nu}{u_\tau^4}, \qquad y^+ = \frac{y u_\tau}{\nu},
$$

where u_τ and ν are the wall shear velocity and the kinematic viscosity of the flow medium, respectively. The profiles of the turbulence intensities from the study of Kim, Moin & Moser (1987) were differentiated to obtain the inhomogeneous part of the dissipation rate. The homogeneous contribution was obtained utilizing the data mentioned above and the ϵ profile.

The data in figure 2 imply that the homogeneous and inhomogeneous parts

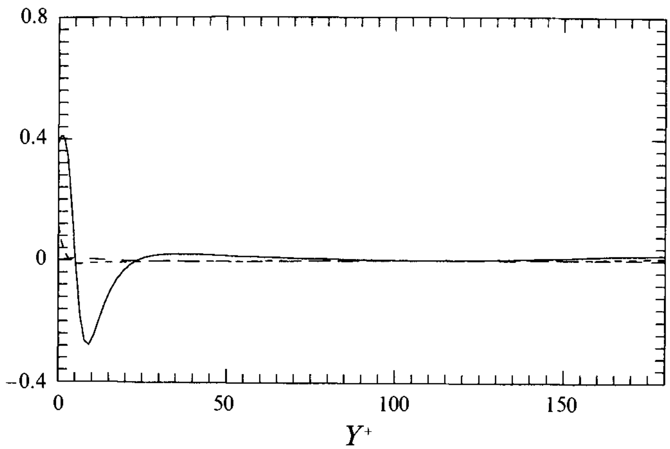

FIGURE 3. Inhomogeneous part of ϵ, contributions of the intensity components: ———, $(\frac{1}{4}\nu\,\partial^2\overline{u_1^2}/\partial x_2^2)/\epsilon$; — — —, $(\frac{1}{4}\nu\,\partial^2\overline{u_2^2}/\partial x_2^2)/\epsilon$; - - - - - -, $(\frac{1}{4}\nu\,\partial^2\overline{u_3^2}/\partial x_2^2)/\epsilon$.

contribute equally to ϵ at the wall. The inhomogeneous part increases the dissipation rate in the region of the viscous sublayer and decreases it in the buffer region. Hence, the steep rise of dissipation rate close to the wall and the secondary peak in its distribution in the buffer region can be explained by the first term of (6.1). This term asymptotically disappears in the logarithmic region. The profile of the homogeneous part of ϵ has an increasing trend in the viscous sublayer, reaches a maximum in the buffer region and decreases thereafter away from the wall region.

In a recent study, Bradshaw & Perot (1993) compared the turbulent dissipation rate ϵ and the true rate of dissipation $\bar{\epsilon}$ in a channel flow using the results of direct numerical simulations. The difference between ϵ and $\bar{\epsilon}$ was found to be less than 2% everywhere across the flow field. In contrast to these findings, there is a substantial difference between the homogeneous and inhomogeneous parts of turbulent dissipation rate in the near-wall region. Figure 3 shows contributions to the inhomogeneous part of ϵ arising from different intensity components. The variations of streamwise component intensity in the viscous sublayer and the buffer region accounts primarily for the important role of the inhomogeneous part of ϵ near the wall.

6.2. *Locally homogeneous turbulence*

In §5, we have explored the assumption of local homogeneity in order to simplify the derived equations. Hence it is appropriate to validate this assumption against the simulation databases. In this subsection, we shall test the applicability of these fundamental ideas using the data of Gilbert & Kleiser (1991).

From the direct numerical simulations, data are currently available only for the terms of the dissipation equation. Consequently, we used the relationships derived in the Appendix to relate these terms to the properties of two-point correlation functions. In this way, we provided the basis for checking the relations used in §5 for the derivation of the simplified equation that governs the homogeneous part of the turbulent dissipation rate.

In the presentations that follow, all terms of the dissipation equation are normalized by the wall variable

$$u_\tau^6/\nu^2,$$

and are plotted against the normalized distance from the wall.

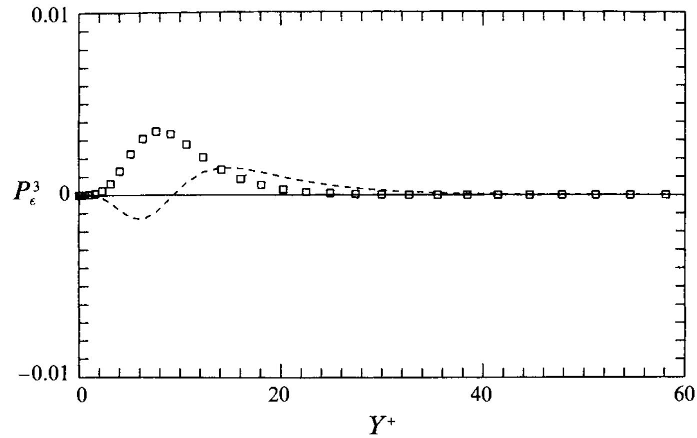

FIGURE 4. Distributions of P_ϵ^3 and the approximation given by (6.4). -------, DNS data; □, equation (6.4).

6.2.1. *Two-point velocity correlations of the second rank*

Using the two-point correlation technique (see Appendix), the gradient production P_ϵ^3 of ϵ can be written as

$$P_\epsilon^3 = -2\nu \overline{u_k \frac{\partial u_i}{\partial x_l}} \frac{\partial^2 U_i}{\partial x_k \partial x_l} = -\nu \frac{\partial}{\partial x_l} \overline{u_i u_k} \frac{\partial^2 U_i}{\partial x_k \partial x_l} + \nu \left(\frac{\partial}{\partial \xi_l} \overline{u_i u_k'} \right)_0 \frac{\partial^2 U_i}{\partial x_k \partial x_l} - \nu \left(\frac{\partial}{\partial \xi_l} \overline{u_k u_i'} \right)_0 \frac{\partial^2 U_i}{\partial x_k \partial x_k}. \tag{6.2}$$

If the assumption of local homogeneity is fulfilled:

$$\left(\frac{\partial}{\partial \xi_l} \overline{u_i u_k'} \right)_0 - \left(\frac{\partial}{\partial \xi_l} \overline{u_k u_i'} \right)_0 \simeq 0, \tag{6.3}$$

then P_ϵ^3 can be approximated as follows:

$$P_\epsilon^3 \simeq -\nu \frac{\partial}{\partial x_l} \overline{u_i u_k} \frac{\partial^2 U_i}{\partial x_k \partial x_l}. \tag{6.4}$$

Figure 4 shows the comparison between the data deduced from the simulated flow field and the approximation given by (6.4). In the viscous sublayer and the buffer region ($y^+ < 35$), the assumption of local homogeneity, (6.3), is not satisfied. Away from the near-wall region ($y^+ > 35$), both terms plotted in figure 4 are approximately equal and decrease towards zero in the logarithmic region. The assumption of local homogeneity, (6.3), is satisfied in the part of the flow that corresponds to the local Reynolds number:

$$R_\lambda = \frac{\lambda q}{\nu} \geqslant 50, \tag{6.5}$$

where λ is the Taylor microscale.

6.2.2. *Two-point pressure/velocity correlations*

In order to determine the applicability of the derived approximations (5.2*d*) and (5.5), we also analysed the two-point pressure/velocity correlations. The pressure

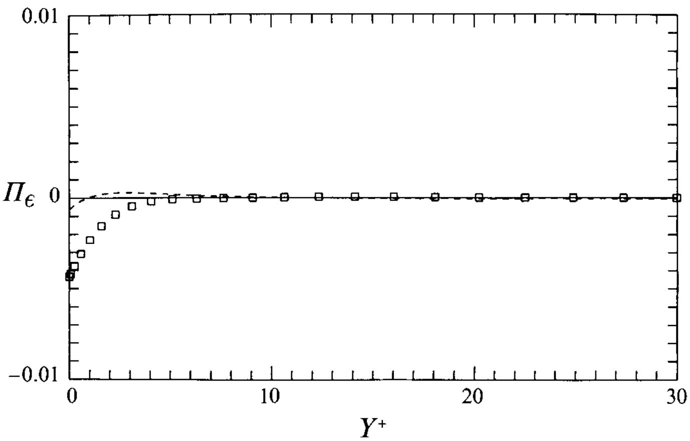

FIGURE 5. Distributions of Π_ϵ and the approximation given by (6.8). -------, DNS data; □, equation (6.8).

term Π_ϵ of the ϵ equation can be expressed as follows (see also Appendix):

$$\Pi_\epsilon = -\frac{2\nu}{\rho}\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial^2 p}{\partial x_i \partial x_l}} = -\frac{\nu}{2\rho}\frac{\partial}{\partial x_i}\Delta_x\overline{u_i p} + \frac{\nu}{2\rho}\frac{\partial}{\partial x_i}[(\Delta_\xi \overline{u_i p'})_0 + (\Delta_\xi \overline{p u_i'})_0] + \frac{\nu}{\rho}\left[\left(\frac{\partial}{\partial \xi_i}\Delta_\xi \overline{u_i p'}\right)_0 - \left(\frac{\partial}{\partial \xi_i}\Delta_\xi \overline{p u_i'}\right)_0\right]. \quad (6.6)$$

If the flow is locally homogeneous

$$(\Delta_\xi \overline{u_i p'})_0 + (\Delta_\xi \overline{p u_i'})_0 \simeq 0, \quad (6.7a)$$

$$\left(\frac{\partial}{\partial \xi_i}\Delta_\xi \overline{u_i p'}\right)_0 - \left(\frac{\partial}{\partial \xi_i}\Delta_\xi \overline{p u_i'}\right)_0 \simeq 0, \quad (6.7b)$$

then Π_ϵ transforms as

$$\Pi_\epsilon \simeq -\frac{\nu}{2\rho}\frac{\partial}{\partial x_i}\Delta_x \overline{u_i p}. \quad (6.8)$$

The form of (6.8) suggests that the derived approximation for Π_ϵ is not closed.

Figure 5 shows a comparison between the data for Π_ϵ obtained from direct numerical simulations and the derived approximation (6.8). These data suggest that the assumption of local homogeneity is not applicable in the region of the viscous sublayer ($y^+ \leqslant 6$) and leads to unrealistic data for the pressure term at the wall. The approximate expression (6.8) for Π_ϵ and the numerical data show that derivatives of two-point pressure/velocity correlations are virtually negligible away from the wall region.

6.2.3. *Two-point velocity correlations of the third rank*

The concept of local homogeneity was also tested for the terms that contain derivatives of two-point velocity correlations of the third rank. Using the two-point correlation technique of § 3, it is possible to show that it is necessary to consider the following correlations:

$$2\frac{\partial}{\partial x_k}\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k u_i}{\partial x_l}} = \frac{1}{2}\frac{\partial}{\partial x_k}\Delta\overline{u_i u_k u_i} - \frac{\partial}{\partial x_k}[(\Delta_\xi \overline{u_i u_k u_i'})_0 + (\Delta_\xi \overline{u_i u_k' u_i'})_0]. \quad (6.9)$$

Since these correlations are not yet available from the simulation databases, we analysed the applicability of the ideas mentioned above for the interpretation of the so-called transport term T_ϵ:

$$T_\epsilon = -\nu \frac{\partial}{\partial x_k} \overline{u_k \frac{\partial u_i}{\partial x_l} \frac{\partial u_i}{\partial x_l}}, \tag{6.10}$$

of the ϵ equation.

The transport term T_ϵ can be interpreted in terms of two-point velocity correlations of the third rank as follows:

$$T_\epsilon = -\frac{\nu}{8} \frac{\partial}{\partial x_k} \Delta_x \overline{u_k u_i u_i} - \nu \frac{\partial}{\partial x_k} [\tfrac{1}{2} (\Delta_\xi \overline{u_i u_i u_k'})_0 - (\Delta_\xi \overline{u_k u_i u_i{}'})_0]. \tag{6.11}$$

Since

$$(\Delta_\xi \overline{u_i u_k u_i'})_0 = (\Delta_\xi \overline{u_i u_i' u_k'})_0, \tag{6.12a}$$

$$(\Delta_\xi \overline{u_i u_i u_k'})_0 = (\Delta_\xi \overline{u_k u_i' u_i'})_0, \tag{6.12b}$$

and assuming that the fine-scale structure of turbulence is locally homogeneous:

$$(\Delta_\xi \overline{u_i u_k u_i'})_0 \simeq -(\Delta_\xi \overline{u_i u_i' u_k'})_0, \tag{6.13a}$$

$$(\Delta_\xi \overline{u_i u_i u_k'})_0 \simeq -(\Delta_\xi \overline{u_k u_i' u_i'})_0, \tag{6.13b}$$

it follows that

$$(\Delta_\xi \overline{u_i u_i u_k'})_0 \simeq (\Delta_\xi \overline{u_k u_i u_i'})_0 \simeq 0. \tag{6.14}$$

Utilizing the derivation given above reduces T_ϵ to the form

$$T_\epsilon \simeq -\frac{\nu}{8} \frac{\partial}{\partial x_k} \Delta_x \overline{u_k u_i u_i}. \tag{6.15}$$

We note that the deduced approximation (6.15) for T_ϵ is not closed.

The simulated data for T_ϵ and the derived approximation (6.15) are shown in figure 6(*a*,*b*). Close to the wall ($y^+ \leqslant 20$), the degree of agreement is slightly better in comparison with the data displayed in figures 4 and 5. However, the numerical data shown in figure 6(*b*) tend towards the derived approximation (6.15) for the transport term slowly, indicating that the assumption of local homogeneity is not strictly satisfied even far away from the near-wall region ($y^+ \geqslant 100$).

The behaviour of the transport term T_ϵ relative to the imbalance in the budget of the ϵ equation is shown in figure 7 on an expanded scale. In the outer part of the flow the imbalance in the dissipation budget (relative to the destruction term Υ) continuously increases and reaches a maximum value of about 17% at the channel centreline. The data in figure 7 also suggest that the transport term T_ϵ and the imbalance in the budget of the ϵ equation are of the same order near the channel centreline. Therefore, we are unable to justify the applicability of the assumption of local homogeneity for the terms of (4.4) that involves two-point velocity correlations of the third rank. More realistic data are therefore required to clarify this important issue precisely. In the meantime, it is safe to retain the term

$$-\frac{\nu}{2} \frac{\partial}{\partial x_k} [(\Delta_\xi \overline{u_s u_k u_s'})_0 + (\Delta_\xi \overline{u_s u_s' u_k'})_0] \tag{6.16}$$

which was disregarded in §5 by assuming that small-scale structure of turbulence is locally homogeneous.

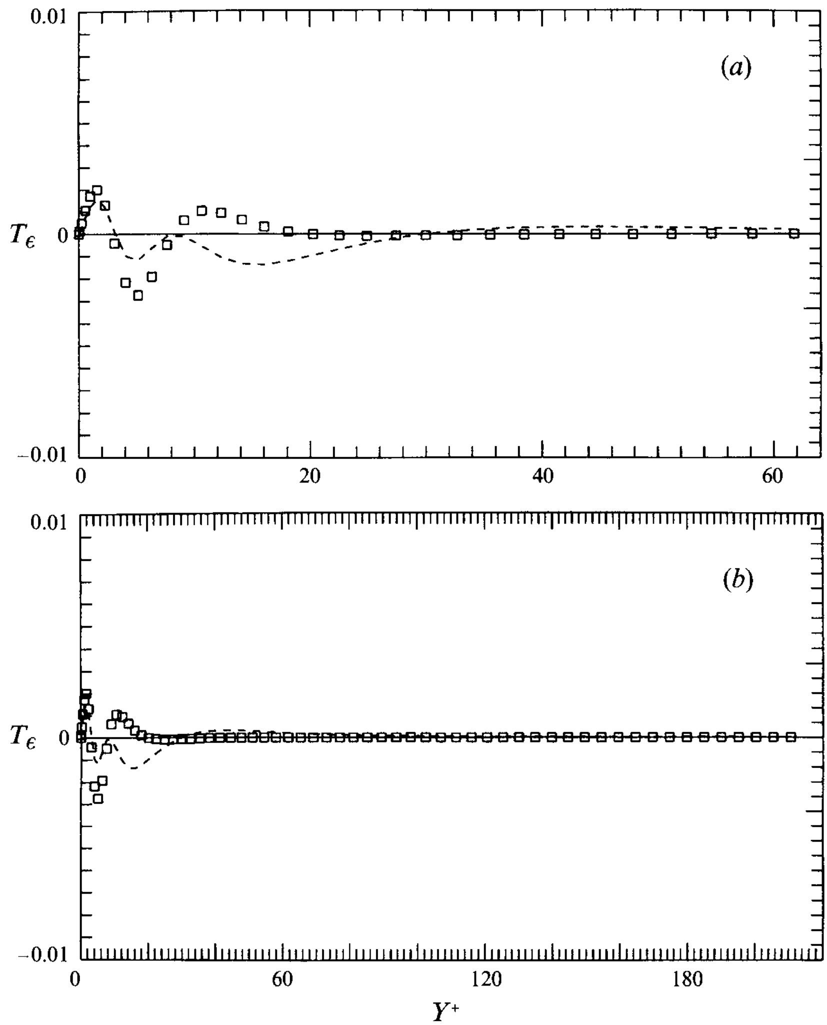

FIGURE 6. Distributions of T_ϵ and the approximation given by (6.15). (*a*) Near-wall region; (*b*) outer flow region. -------, DNS data; □, equation (6.15).

6.3. *The structure of the approximate equation for* $-\nu(\Delta_\xi \overline{u_s u_s'})_0$

In order to study the balance of the approximate equation that governs the homogeneous part of ϵ, we shall first write the revised form of (5.6) by retaining the derivatives of two-point velocity correlations of third rank:

$$\begin{aligned}
&-\nu\frac{\partial}{\partial t}(\Delta_\xi \overline{u_s u_s'})_0 - \nu U_k \frac{\partial}{\partial x_k}(\Delta_\xi \overline{u_s u_s'})_0 - 2\nu(\Delta_\xi \overline{u_k u_s'})_0 \frac{\partial U_s}{\partial x_k} \\
&-\frac{\nu}{4}\left[2\overline{u_s u_k}\Delta_x \frac{\partial U_s}{\partial x_k} + (\Delta_x U_k)\frac{\partial}{\partial x_k}\overline{u_s u_s}\right] - 2\nu\left(\frac{\partial^2}{\partial \xi_l \partial \xi_k}\overline{u_s u_s'}\right)_0 \frac{\partial U_k}{\partial x_l} \\
&-\frac{\nu}{2}\frac{\partial}{\partial x_k}\left[(\Delta_\xi \overline{u_s u_k u_s'})_0 + (\Delta_\xi \overline{u_s u_s' u_k'})_0\right] - \nu\left[\Delta_\xi \frac{\partial}{\partial \xi_k}(\overline{u_s u_s' u_k'} - \overline{u_s u_k u_s'})\right]_0 \\
&+\tfrac{1}{2}\nu^2 \Delta_x (\Delta_\xi \overline{u_s u_s'})_0 + 2\nu^2(\Delta_\xi \Delta_\xi \overline{u_s u_s'})_0 \simeq 0. \qquad (6.17)
\end{aligned}$$

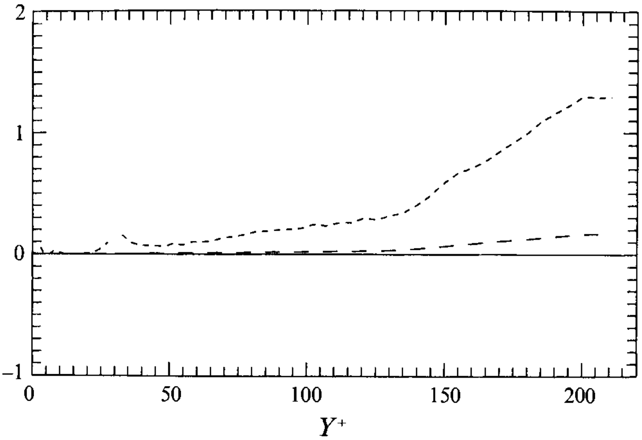

FIGURE 7. Distribution of the imbalance in the ϵ equation relative to T_ϵ and Υ. - - - - - - -, $|\,\mathrm{Sum}\,|\,/T_\epsilon$; ———, $|\,\mathrm{Sum}\,|\,/\Upsilon$;

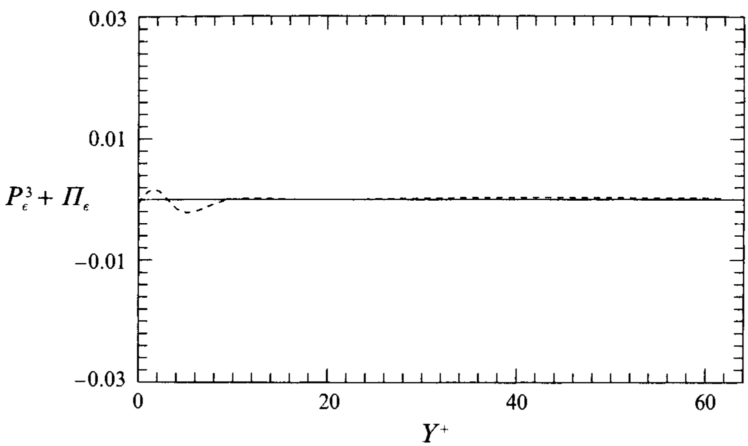

FIGURE 8. Sum of small terms P_ϵ^3 and Π_ϵ obtained from the turbulent channel flow data of Gilbert & Kleiser (1991).

By including the term (6.16) we ensured that the balance of (6.17) is satisfied far away from the wall.

The analysis of the simulated data, in general, contradicts the applicability of the assumption of local homogeneity in the near-wall region. However, the terms considered in the previous section, P_ϵ^3 and Π_ϵ (as well as their sum), are small, as shown in figure 8. Therefore, the error in the approximations introduced for these terms in § 5 might not be large enough to affect the balance of (6.17) significantly close to the wall.

To form the balance of (6.17), the terms of the ϵ equation were re-evaluated from the simulated data using derivations given in the Appendix. Figure 9 (*a,b*) shows the budget of (6.17) computed from the data of Mansour *et al.* (1987). Away from the near-wall region ($y^+ \geqslant 20$), the balance of (6.17) is satisfied to a comparable degree of accuracy as for the ϵ equation. The imbalance in the data increases as the wall is approached and reaches its maximum at the wall. In the region ($2 \leqslant y^+ \leqslant 20$), the

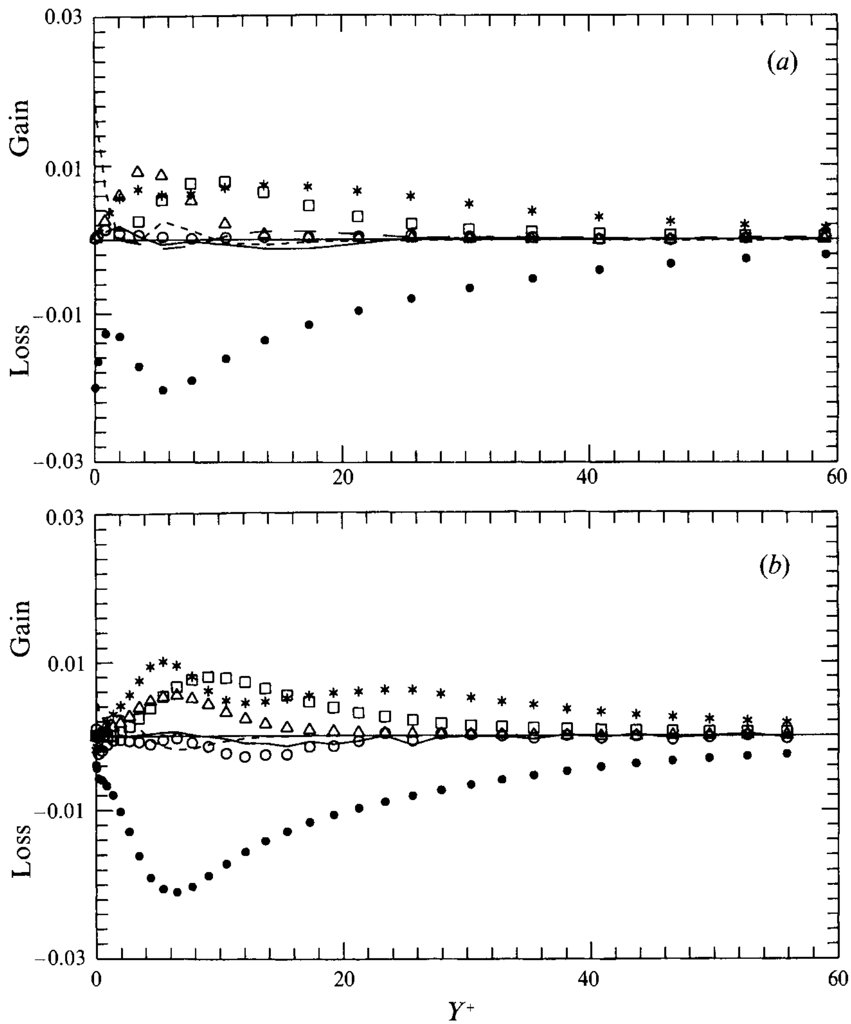

FIGURE 9. Terms in the budget of (6.17) deduced from the simulated data of Mansour *et al.* (1987). (*a*) Budget of ϵ equation: $\triangle$, P_ϵ^1; $\square$, P_ϵ^2; — — — — —, P_ϵ^3; $*$, P_ϵ^4; ———, $T_\epsilon + \Pi_\epsilon$; - - - - - -, D_ϵ; $\bullet$, $-\Upsilon$; $\circ$, sum of all terms; (*b*) budget of (6.17): $\triangle$, $T_1 = -2\nu(\Delta_\xi \overline{u_k u_s'})_0 \partial U_s/\partial x_k$; ———, $T_2 = -\frac{1}{4}\nu[2\overline{u_s u_k}\Delta_x \partial U_s/\partial x_k + (\Delta_x U_k)(\partial/\partial x_k)\overline{u_s u_s}]$; $\square$, $T_3 = -2\nu[(\partial^2/\partial\xi_l\partial\xi_k)\overline{u_s u_s'}]_0 \partial U_k/\partial x_l$; $*$, $T_4 = -\frac{1}{2}\nu(\partial/\partial x_k)[(\Delta_\xi \overline{u_s u_k u_s'})_0 + (\Delta_\xi \overline{u_s u_s' u_k'})_0] + \nu[\Delta_\xi(\partial/\partial\xi_k)(\overline{u_s u_s' u_k'} - \overline{u_s u_k u_s'})]_0$; - - - - - -, $T_5 = \frac{1}{2}\nu^2\Delta_x(\Delta_\xi \overline{u_s u_s'})_0$; $\bullet$, $T_6 = 2\nu^2(\Delta_\xi\Delta_\xi \overline{u_s u_s'})_0$; $\circ$, sum of all terms.

relative imbalance in the data is less than 10% of the destruction term $2\nu^2(\Delta_\xi\Delta_\xi\overline{u_s u_s'})_0$.

The imbalance shown in figure 9(*b*) in the region adjacent to the wall ($y^+ \leqslant 2$) is not a matter of serious concern, since the solution of (6.17) can be deduced using kinematic considerations only. Using the two-point correlation technique, the authors have shown (Jovanović *et al.* 1992) that the turbulent dissipation rate can be interpreted in terms of the Taylor microscale as follows:

$$\epsilon = \tfrac{1}{4}\nu\Delta_x q^2 + 5\nu\tfrac{q^2}{\lambda^2}. \qquad (6.18)$$

Expanding the instantaneous velocity field in a Taylor series around the wall values

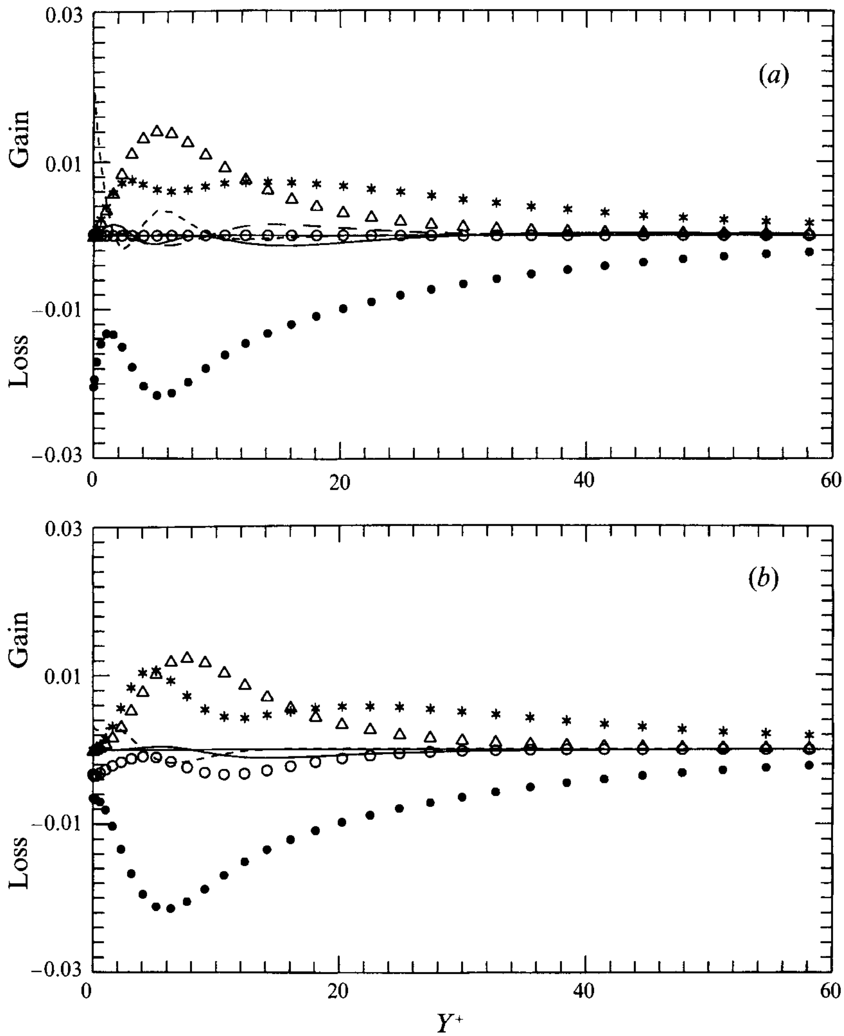

FIGURE 10. Terms in the budget of (6.17) deduced from the simulated data of Gilbert & Kleiser (1991). (*a*) Budget of ϵ equation: $\triangle$, $P^1_\epsilon + P^2_\epsilon$; — — — — — —, P^3_ϵ; $*$, P^4_ϵ; ———, T_ϵ; — —— ——, Π_ϵ; - - - - - - - -, D_ϵ; $\bullet$, $-\Upsilon$; $\circ$, sum of all terms; (*b*) budget of (6.17): $\triangle$, $T_1 + T_3 = -2\nu(\Delta_\xi \overline{u_k u'_s})_0 \partial U_s/\partial x_k - 2\nu[(\partial^2/\partial\xi_l\partial\xi_k)\overline{u_s u'_s}]_0 \partial U_k/\partial x_l$; ———, $T_2 = -\frac{1}{4}\nu[2\overline{u_s u_k}\Delta_x \partial U_s/\partial x_k + (\Delta_x U_k)(\partial/\partial x_k)\overline{u_s u_s}]$; $*$, $T_4 = -\frac{1}{2}\nu(\partial/\partial x_k)[(\Delta_\xi \overline{u_s u_k u'_s})_0 + (\Delta_\xi \overline{u_s u'_s u'_k})_0] + \nu[\Delta_\xi(\partial/\partial\xi_k)(\overline{u_s u'_s u'_k} - \overline{u_s u_k u'_s})]_0$; - - - - - - - -, $T_5 = \frac{1}{2}\nu^2\Delta_x(\Delta_\xi \overline{u_s u'_s})_0$; $\bullet$, $T_6 = 2\nu^2(\Delta_\xi\Delta_\xi \overline{u_s u'_s})_0$; $\circ$, sum of all terms.

($x_2 = 0$), one confirms that λ is a linear function of the distance from the wall:

$$\lambda = \sqrt{10}x_2. \tag{6.19}$$

Therefore, owing to the explicit form of (6.19), it is possible to avoid integration of (6.17) in the vicinity of the wall.

We also used the data of Gilbert & Kleiser (1991) to validate the balance of (6.17). The results given in figure 10(*a*,*b*) confirm that the derived equation holds away from the near-wall region. The behaviour of the data close to the wall is less encouraging compared with the data shown in figure 9(*b*).

7. Discussion

It is of interest to analyse the relevance of the data presented to the development of turbulence closure techniques and methods for the interpretation of experimental data. In what follows, we shall illustrate the impact of the deduced results on the determination of the turbulent dissipation rate close to the wall.

7.1. *Implications for turbulence closure*

As representative data that illustrate current trends in developments of turbulence closure, we shall analyse the predictions of turbulent channel flow at low Reynolds number of Hanjalić *et al.* (1992). The data were obtained using a state of the art second-moment closure with the equation for ϵ as follows:

$$\frac{\partial \epsilon}{\partial t} + U_k \frac{\partial \epsilon}{\partial x_k} = -C_{\epsilon 1} \frac{\epsilon}{k} \overline{u_i u_k} \frac{\partial U_i}{\partial x_k} - C_{\epsilon 2} f_\epsilon \frac{\epsilon \tilde{\epsilon}}{k} + C_{\epsilon 3} \nu \frac{k}{\epsilon} \overline{u_j u_k} \frac{\partial^2 U_i}{\partial x_j \partial x_l} \frac{\partial^2 U_i}{\partial x_k \partial x_l} + \frac{\partial}{\partial x_k} \left(C_\epsilon \frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial \epsilon}{\partial x_l} \right) + \nu \Delta_x \epsilon, \tag{7.1}$$

where

$$\left. \begin{array}{ll} k = \frac{1}{2}\overline{u_s u_s}, & f_\epsilon = 1 - \frac{0.52}{1.92}\{\exp[-(\frac{1}{6}R_{e_t})^2]\}, \\ R_{e_t} = k^2/\nu\epsilon, & \tilde{\epsilon} = \epsilon - 2\nu(\partial k^{1/2}/\partial x_2)^2, \\ C_\epsilon = 0.18, & C_{\epsilon 1} = 1.44, \\ C_{\epsilon 2} = 1.92, & C_{\epsilon 3} = 0.5. \end{array} \right\} \tag{7.2}$$

The first term on the right-hand side of (7.1) approximates the production by the mean velocity gradient (P_ϵ^1) and the mixed production (P_ϵ^2) of the dissipation equation (A4). The firm analytical proof for the closure of these terms (for the homogeneous parts) is given in the paper by Chou (1945).

The second term represents the difference of the turbulent production (P_ϵ^4) and viscous destruction (Υ), the two dominant terms in the balance of the dissipation equation. Following a practice first introduced by Davydov (1961), these terms are exclusively determined from the laws of the decay of grid turbulence. As shown by Hanjalić & Launder (1976), the closure deduced from grid turbulence requires further semi-empirical modifications for near-wall applications.

The third term is assigned as an approximation for the gradient production (P_ϵ^3) of the dissipation equation. The fourth term accounts for the diffusive transport (T_ϵ) of ϵ. The adopted closure for this term is analogous to the form used for the interpretation of the similar term in the equation for kinetic energy of turbulence. The last term is the viscous diffusion.

Lumley (1978) and the follow-up contributions (e.g. Tselepidakis 1991; Rodi & Mansour 1993; Kessler 1993) to models for the dissipation rate equation would argue that the modern models are for the entire right-hand side of (A4) and not the individual terms in the dissipation rate equation.

The predicted profile of ϵ obtained from (7.1) is displayed in figure 11 together with the results of similar calculations from the study of Mansour, Kim & Moin (1989). The latter data were obtained by solving the simplified form of (7.1) taking $C_{\epsilon 3} = 0$. Note that the data in figure 11 correspond to flow conditions identical to Kim *et al.* (1987).

The predicted data in figure 11 resemble the shape of the homogeneous part of ϵ across the entire flow. This conclusion holds particularly for the profile obtained from the simplified form of (7.1) with $C_{\epsilon 3} = 0$. The ϵ profiles are smooth, with no

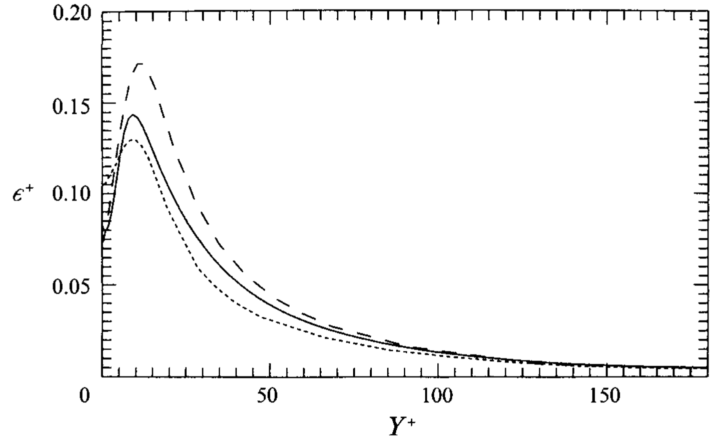

FIGURE 11. Computed distributions of ϵ across the channel. ———, ϵ_h Mansour *et al.* (1987); — — — —, Hanjalić *et al.* (1992); - - - - - - -, Mansour *et al.* (1989).

evidence of a secondary peak close to the wall. Therefore, the predicted data from figure 11 imply that *current procedures for the determination of ϵ are able to reproduce the homogeneous part of the turbulent dissipation rate.* This conclusion inferred from figure 11 is in agreement with the results of the analysis carried out in § 3 and the data shown in figure 11.

Without considering the closure of (6.17) in detail, from the data shown in figure 11 and the analytical results in §§ 3 and 5, a suitable form of equations for the determination of ϵ immediately emerges. Equation (6.1) can be written in the form

$$\epsilon = \tfrac{1}{2}\nu\Delta_x k + \epsilon_h. \tag{7.3}$$

By comparing (6.17) and (7.1) and taking into account the data shown in figure 11, one obtains the equation for the homogeneous part of ϵ:

$$\frac{\partial \epsilon_h}{\partial t} + U_k\frac{\partial \epsilon_h}{\partial x_k} = -C_{\epsilon 1}\frac{\epsilon_h\overline{u_i u_k}}{k}\frac{\partial U_i}{\partial x_k} - C_{\epsilon 2}f_\epsilon\frac{\epsilon_h\tilde{\epsilon}_h}{k} + \frac{\partial}{\partial x_k}\left(C_\epsilon\frac{k}{\epsilon_h}\overline{u_k u_l}\frac{\partial \epsilon_h}{\partial x_l}\right) + \frac{\nu}{2}\Delta_x\epsilon_h, \tag{7.4}$$

where

$$\tilde{\epsilon}_h = \epsilon_h - \nu\left(\frac{\partial k^{1/2}}{\partial x_2}\right)^2, \qquad R_{e_t} = \frac{k^2}{\nu\epsilon_h}. \tag{7.5}$$

In formulating (7.4), we neglected the terms

$$-\tfrac{1}{4}\left[2\overline{u_s u_k}\Delta_x\tfrac{\partial U_s}{\partial x_k} + (\Delta_x U_k)\tfrac{\partial}{\partial x_k}\overline{u_s u_s}\right] \tag{7.6}$$

of (6.17), since these are very small in wall-bounded flows. We also ignored the term

$$\nu\frac{k}{\epsilon}\overline{u_j u_k}\frac{\partial^2 U_i}{\partial x_j \partial x_l}\frac{\partial^2 U_i}{\partial x_k \partial x_l}, \tag{7.7}$$

of (7.1), since the formulated closure for this term is not very reliable. There is no firm theoretical justification for the form suggested, which is quadratic in the second derivative of the mean velocity. The closure for this term has additional shortcomings: (i) the predicted magnitude of P_ϵ^3 is significantly higher than numerical data; (ii) while the numerical results indicate that P_ϵ^3 changes sign in the near-wall region, the closure approximation implies that this term is always positive; (iii) the

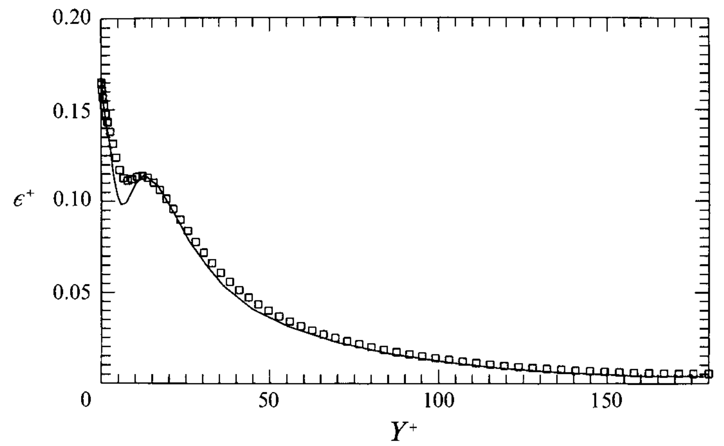

FIGURE 12. Comparison of the turbulent dissipation rate from Mansour *et al.* (1987) and the numerical predictions of (7.3) and (7.4). ———, (7.3) and (7.4); □, DNS data.

numerical data and the results in §6 show that P_ϵ^3 is small and has no effect on the budget of the homogeneous part of ϵ even very close to the wall. We have neglected the term P_ϵ^3 entirely in formulating the modified equations (7.3) and (7.4) for prediction of the turbulent dissipation rate in wall-bounded flows. The other terms in (7.4) are clearly analogous to the corresponding terms of (6.17). In view of the results presented in §6, *the transport term in* (7.4) *may be also interpreted statistically as a measure of the departure from the local homogeneity of turbulence at small scales.*

Equations (7.3) and (7.4) were integrated numerically by Hanjalić & Jakirlić (1993) using the mean velocity, kinetic energy of turbulence and shear stress profiles from the simulated data (Kim *et al.* 1987). The boundary conditions used were

$$\epsilon_h = \frac{\nu}{2}\frac{\partial^2 k}{\partial x_2^2}, \tag{7.8}$$

at the wall and

$$\frac{\partial \epsilon_h}{\partial x_2} = 0, \tag{7.9}$$

at the channel centreline. Figure 12 shows the predictions of ϵ against the profile deduced from the simulated flow field. The degree of agreement achieved demonstrates the necessity of including the inhomogeneous part of ϵ for obtaining improved data predictions in the region close to the wall.

7.2. *Analysis of measurements of the turbulent dissipation rate*

In connection with the issues discussed above, we shall analyse turbulence measurements in a fully developed pipe flow from Laufer (1953). In the past, his results were used to support the closure for the ϵ equation.

Apart from turbulence intensities, Laufer also measured the distribution of ϵ in the near-wall region. The energy dissipation rate was inferred from hot-wire measurements of the velocity derivatives. For this purpose, Laufer applied the Taylor hypothesis to determine derivatives in the streamwise direction and isotropic relationships to estimate derivatives of radial and tangential velocity components.

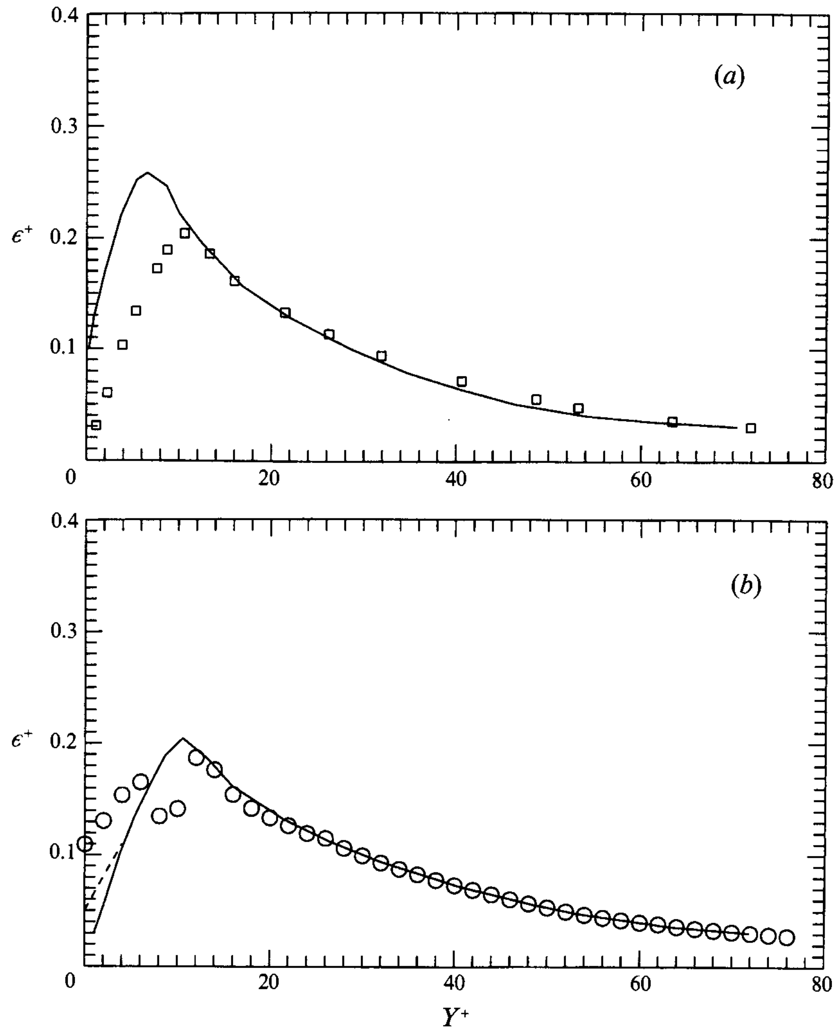

FIGURE 13. Dissipation rate measurements in pipe flow from Laufer (1953). (*a*) Original data □ (and corrections) (———) introduced by Townsend (1976); (*b*) ○, re-evaluated data using the inhomogeneous term of (6.1); ———, original data; -------, corrected data.

Only two of the nine terms of (3.10) were measured directly by the application of the two-point correlation technique. The derivatives of the streamwise velocity component in the radial and tangential directions were obtained from the slope of the straight line of the plot of the cross-correlation coefficient measured by the two separated hot-wire probes versus the separation distance squared. Figure 13(*a*) shows Laufer's original data together with corrections introduced by Townsend (1976) to bring these results into accordance with the measured slopes of the turbulence intensities in the viscous sublayer.

On examining the measurement procedure used by Laufer, it is more appropriate to assign his original data to the homogeneous part of the dissipation rate. As figure 13(*b*) shows, only marginal correction of the original data is required to bring these data into agreement with the measured profiles of turbulence intensities close to the wall. The total dissipation rate ϵ determined from these data using the inhomogeneous part of (6.1) also exhibits a similar shape near the wall to the profile shown in figure 2 obtained from direct numerical simulation. Note that the local peak in the ϵ distribution close to the wall is also reproduced from the experimental data.

8. Conclusions

Statistical analysis based on the two-point correlation technique shows that the turbulent dissipation rate can be decomposed into homogeneous and inhomogeneous parts. The inhomogeneous part is especially important in wall-bounded flows, since it contributes half of the dissipation rate at the wall.

Away from the wall, the derivatives of two-point velocity correlations of second rank and the derivatives of two-point pressure/velocity correlations that contribute to the balance of the equation for the homogeneous part of ϵ are consistent with the local homogeneity assumption. However, the same conclusion does not hold for the derivatives of two-point velocity correlations of third rank. For the derivatives of triple correlations, the local homogeneity assumption does not seem to be applicable across the entire flow. The analysed data also show that the concept of locally homogeneous turbulence does not hold in the near-wall region. Using the simulated data of turbulent channel flow, it has been shown that, in spite of all deficiencies mentioned above, the approximate equation (6.17) for the homogeneous part of ϵ balances the data reasonably well across the entire flow field.

The comparisons of the predicted profiles of ϵ for turbulent channel flow with simulated data imply that *current prediction procedures are able to reproduce the homogeneous part of* ϵ. Simple modifications of existing closures for the ϵ equation were proposed. The predictions of the modified equations agree favourably with the data of direct numerical simulations.

We analysed Laufer's (1953) measurements of the turbulent dissipation rate close to the wall. By adding the inhomogeneous part of ϵ to the measured data, the behaviour of the dissipation rate was obtained, resembling the corresponding profile obtained from direct numerical simulations. The re-evaluated distribution of ϵ also shows a secondary peak close to the wall.

The authors are grateful to Dr R. Kessler of DLR-Göttingen for providing the direct simulation data for turbulent channel flow and to Mr Mark Benak who undertook, as many times before, refinement of the text. We gratefully acknowledge the support (Jo 240/1-1) given to us by the Deutsche Forschungsgemeinschaft (DFG).

Appendix A. Application of the two-point correlation technique for interpreting the equation for the dissipation rate

We shall now analyse the advantages of (4.4) over the corresponding equation that defines the trace ϵ:

$$\epsilon = \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k}}$$

of the dissipation tensor ϵ_{ij}.

To derive the equation for the dissipation rate, ϵ, we first differentiate (2.1) for the instantaneous fluctuation with $\partial/\partial x_l$ to obtain

$$\frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_l}+\frac{\partial U_k}{\partial x_l}\frac{\partial u_i}{\partial x_k}+U_k\frac{\partial^2 u_i}{\partial x_k \partial x_l}+\frac{\partial u_k}{\partial x_l}\frac{\partial U_i}{\partial x_k}+u_k\frac{\partial^2 U_i}{\partial x_k \partial x_l}$$
$$+\frac{\partial^2}{\partial x_k \partial x_l}(u_k u_i-\overline{u_k u_i})=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_l}+\nu\frac{\partial}{\partial x_l}\Delta_x u_i. \quad \text{(A 1)}$$

Multiplying the above equation by $2\nu\partial u_i/\partial x_l$ and using the mass conservation law results in the following equation:

$$\frac{\partial}{\partial t}\nu\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}+2\nu\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial U_k}{\partial x_l}+U_k\frac{\partial}{\partial x_k}\nu\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}+2\nu\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}\frac{\partial U_i}{\partial x_k}$$
$$+\ 2\nu u_k\frac{\partial u_i}{\partial x_l}\frac{\partial^2 U_i}{\partial x_k \partial x_l}+2\nu\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial x_k}+2\nu u_k\frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_i}{\partial x_k \partial x_l}--2\nu\frac{\partial u_i}{\partial x_l}\frac{\partial^2}{\partial x_k \partial x_l}\overline{u_k u_i}$$
$$=-\frac{2\nu}{\rho}\frac{\partial u_i}{\partial x_l}\frac{\partial^2 p}{\partial x_i \partial x_l}+2\nu^2\frac{\partial u_i}{\partial x_l}\Delta_x\frac{\partial u_i}{\partial x_l}. \quad \text{(A 2)}$$

Using the following transformations:

$$2\nu u_k\frac{\partial u_i}{\partial x_l}\frac{\partial^2 u_i}{\partial x_k \partial x_l}=\nu\frac{\partial}{\partial x_k}\left(u_k\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}\right), \quad \text{(A 3a)}$$

$$2\nu^2\frac{\partial u_i}{\partial x_l}\Delta_x\frac{\partial u_i}{\partial x_l}=\nu\Delta_x\nu\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}-2\nu^2\frac{\partial^2 u_i}{\partial x_l \partial x_n}\frac{\partial^2 u_i}{\partial x_l \partial x_n}, \quad \text{(A 3b)}$$

and averaging (A2), we obtain the equation for determining the dissipation rate ϵ:

$$\underset{[1]}{\frac{\partial \epsilon}{\partial t}}+\underset{[2]}{U_k\frac{\partial \epsilon}{\partial x_k}}=\underset{[3]}{-2\nu\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}}\frac{\partial U_i}{\partial x_k}}\underset{[4]}{-2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}}\frac{\partial U_k}{\partial x_l}}\underset{[5]}{-2\nu\overline{u_k\frac{\partial u_i}{\partial x_l}}\frac{\partial^2 U_i}{\partial x_k \partial x_l}}$$

$$\underset{[6]}{-2\nu\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial x_k}}}\underset{[7]}{-\nu\frac{\partial}{\partial x_k}[\overline{u_k\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}}]}\underset{[8]}{-\frac{2\nu}{\rho}\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial^2 p}{\partial x_i \partial x_l}}}\underset{[9]}{-2\nu^2\overline{\frac{\partial^2 u_i}{\partial x_l \partial x_n}\frac{\partial^2 u_i}{\partial x_l \partial x_n}}}\underset{[10]}{+\nu\Delta_x\epsilon}. \quad \text{(A 4)}$$

To gain insight into the significance of the various terms and into the structure of the dissipation equation, we shall apply the two-point correlation technique which was already discussed in § 3. The individual terms of (A4) read as follows:

Term 1

$$\frac{\partial \epsilon}{\partial t}=\frac{\nu}{4}\Delta_x\frac{\partial q^2}{\partial t}-\nu\frac{\partial}{\partial t}(\Delta_\xi\overline{u_i u_i'})_0, \quad \text{(A 5)}$$

Term 2

$$U_k\frac{\partial \epsilon}{\partial x_k}=\frac{\nu}{4}U_k\frac{\partial}{\partial x_k}\Delta_x q^2-\nu U_k\frac{\partial}{\partial x_k}(\Delta_\xi\overline{u_i u_i'})_0, \quad \text{(A 6)}$$

Term 3

$$P_\epsilon^1 = -2\nu\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}}\frac{\partial U_i}{\partial x_k} = -\frac{\nu}{2}\Delta_x\overline{u_iu_k}\frac{\partial U_i}{\partial x_k} + \nu\left[(\Delta_\xi\overline{u_ku_i'})_0\frac{\partial U_i}{\partial x_k} + (\Delta_\xi\overline{u_iu_k'})_0\frac{\partial U_i}{\partial x_k}\right], \quad \text{(A 7)}$$

Term 4

$$P_\epsilon^2 = -2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}}\frac{\partial U_k}{\partial x_l} = -\frac{\nu}{2}\frac{\partial^2 q^2}{\partial x_k\partial x_l}\frac{\partial U_k}{\partial x_l} + 2\nu(\frac{\partial^2}{\partial \xi_k\partial \xi_l}\overline{u_iu_i'})_0\frac{\partial U_k}{\partial x_l}, \quad \text{(A 8)}$$

Term 5

$$P_\epsilon^3 = -2\nu\overline{u_k\frac{\partial u_i}{\partial x_l}}\frac{\partial^2 U_i}{\partial x_k\partial x_l} = -\nu\frac{\partial}{\partial x_l}\overline{u_iu_k}\frac{\partial^2 U_i}{\partial x_k\partial x_l} + \nu\left(\frac{\partial}{\partial \xi_l}\overline{u_iu_k'}\right)_0\frac{\partial^2 U_i}{\partial x_k\partial x_l}$$
$$-\nu\left(\frac{\partial}{\partial \xi_l}\overline{u_ku_i'}\right)_0\frac{\partial^2 U_i}{\partial x_k\partial x_l}, \quad \text{(A 9)}$$

Terms 6 and 7

$$P_\epsilon^4 + T_\epsilon = -2\nu\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial x_k}} - \nu\frac{\partial}{\partial x_k}\overline{u_k\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}} = -\frac{\nu}{4}\frac{\partial}{\partial x_k}\Delta_x\overline{u_iu_ku_i}$$
$$+ \frac{\nu}{2}\frac{\partial}{\partial x_k}[(\Delta_\xi\overline{u_iu_ku_i'})_0 + (\Delta_\xi\overline{u_iu_k'u_i'})_0] + \nu\left[\left(\frac{\partial}{\partial \xi_k}\Delta_\xi\overline{u_iu_k'u_i'}\right)_0 - \left(\frac{\partial}{\partial \xi_k}\Delta_\xi\overline{u_iu_ku_i'}\right)_0\right], \quad \text{(A 10)}$$

Term 8

$$\Pi_\epsilon = -\frac{2\nu}{\rho}\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial^2 p}{\partial x_i\partial x_l}} = -\frac{\nu}{2\rho}\frac{\partial}{\partial x_i}\Delta_x\overline{u_ip} + \frac{\nu}{2\rho}\frac{\partial}{\partial x_i}[(\Delta_\xi\overline{u_ip'})_0 + (\Delta_\xi\overline{pu_i'})_0]$$
$$+\frac{\nu}{\rho}\left[\left(\frac{\partial}{\partial \xi_i}\Delta_\xi\overline{u_ip'}\right)_0 - \left(\frac{\partial}{\partial \xi_i}\Delta_\xi\overline{pu_i'}\right)_0\right], \quad \text{(A 11)}$$

Term 9

$$-\Upsilon = -2\nu^2\overline{\frac{\partial^2 u_i}{\partial x_l\partial x_n}\frac{\partial^2 u_i}{\partial x_l\partial x_n}} = -\frac{\nu^2}{8}\Delta_x\Delta_x\overline{u_iu_i} + \nu^2\Delta_x(\Delta_\xi\overline{u_iu_i'})_0 - 2\nu^2(\Delta_\xi\Delta_\xi\overline{u_iu_i'})_0, \quad \text{(A 12)}$$

Term 10

$$D_\epsilon = \nu\Delta_x\epsilon = \frac{\nu^2}{4}\Delta_x\Delta_x\overline{u_iu_i} - \nu^2\Delta_x(\Delta_\xi\overline{u_iu_i'})_0. \quad \text{(A 13)}$$

By adding all of the derived terms defined by (A5)–(A13), we obtain the following form of the dissipation equation:

$$\frac{\nu}{4}\Delta_x\frac{\partial q^2}{\partial t} - \nu\frac{\partial}{\partial t}(\Delta_\xi\overline{u_iu_i'})_0 + \frac{\nu}{4}U_k\frac{\partial}{\partial x_k}\Delta_x q^2 - \nu U_k\frac{\partial}{\partial x_k}(\Delta_\xi\overline{u_iu_i'})_0 =$$
$$-\frac{\nu}{2}\Delta_x\overline{u_iu_k}\frac{\partial U_i}{\partial x_k} + \nu(\Delta_\xi\overline{u_ku_i'})_0\frac{\partial U_i}{\partial x_k} + \nu(\Delta_\xi\overline{u_iu_k'})_0\frac{\partial U_i}{\partial x_k} - \frac{\nu}{2}\frac{\partial^2 q^2}{\partial x_k\partial x_l}\frac{\partial U_k}{\partial x_l}$$
$$+ 2\nu\left(\frac{\partial^2}{\partial \xi_k\partial \xi_l}\overline{u_iu_i'}\right)_0\frac{\partial U_k}{\partial x_l} - \nu\frac{\partial}{\partial x_l}\overline{u_iu_k}\frac{\partial^2 U_i}{\partial x_k\partial x_l} + \nu\left(\frac{\partial}{\partial \xi_l}\overline{u_iu_k'}\right)_0\frac{\partial^2 U_i}{\partial x_k\partial x_l}$$
$$-\nu\left(\frac{\partial}{\partial \xi_l}\overline{u_ku_i'}\right)_0\frac{\partial^2 U_i}{\partial x_k\partial x_l} - \frac{\nu}{4}\frac{\partial}{\partial x_k}\Delta_x\overline{u_iu_ku_i} + \frac{\nu}{2}\frac{\partial}{\partial x_k}[(\Delta_\xi\overline{u_iu_ku_i'})_0 + (\Delta_\xi\overline{u_iu_k'u_i'})_0]$$
$$+ \nu\left[\left(\frac{\partial}{\partial \xi_k}\Delta_\xi\overline{u_iu_k'u_i'}\right)_0 - \left(\frac{\partial}{\partial \xi_k}\Delta_\xi\overline{u_iu_ku_i'}\right)_0\right] - \frac{\nu}{2\rho}\frac{\partial}{\partial x_i}\Delta_x\overline{u_ip} + \frac{\nu}{2\rho}\frac{\partial}{\partial x_i}[(\Delta_\xi\overline{pu_i'})_0$$

$$+ (\Delta_\xi \overline{u_i p'})_0] + \frac{\nu}{\rho}\left[\left(\frac{\partial}{\partial \xi_i}\Delta_\xi \overline{u_i p'}\right)_0 - \left(\frac{\partial}{\partial \xi_i}\Delta_\xi \overline{p u_i'}\right)_0\right] - \frac{\nu^2}{8}\Delta_x \Delta_x \overline{u_i u_i}$$
$$+ \nu^2 \Delta_x (\Delta_\xi \overline{u_i u_i'})_0 - 2\nu^2 (\Delta_\xi \Delta_\xi \overline{u_i u_i'})_0 + \frac{\nu^2}{4}\Delta_x \Delta_x \overline{u_i u_i} - \nu^2 \Delta_x (\Delta_\xi \overline{u_i u_i'})_0 \qquad \text{(A 14)}$$

Equation (A14) is composed of the terms that are included in (4.4) and higher-order terms. The higher-order terms are directly related to the equation for q^2. To illustrate this point further, we first contract (2.3) to obtain

$$\frac{\partial q^2}{\partial t} + U_k \frac{\partial q^2}{\partial x_k} + 2\overline{u_i u_k}\frac{\partial U_i}{\partial x_k} + \frac{\partial \overline{u_i u_i u_k}}{\partial x_k} + \frac{2}{\rho}\overline{u_i \frac{\partial p}{\partial x_i}} + 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} - \nu \Delta_x q^2 = 0. \qquad \text{(A 15)}$$

By applying the Laplace operator Δ_x to (A15) and multiplying the resulting equation by $\nu/4$, we obtain

$$\frac{\nu}{4}\Delta_x \frac{\partial q^2}{\partial t} + \frac{\nu}{2}\frac{\partial U_k}{\partial x_l}\frac{\partial q^2}{\partial x_l \partial x_k} + \frac{\nu}{4}U_k \frac{\partial}{\partial x_k}\Delta_x q^2 + \frac{\nu}{4}\frac{\partial q^2}{\partial x_k}\Delta_x U_k + \nu \frac{\partial \overline{u_i u_k}}{\partial x_l}\frac{\partial^2 U_i}{\partial x_k \partial x_l}$$
$$+ \frac{\nu}{2}(\Delta_x \overline{u_i u_k})\frac{\partial U_i}{\partial x_k} + \frac{\nu}{2}\overline{u_i u_k}\frac{\partial}{\partial x_k}\Delta_x U_i + \frac{\nu}{4}\frac{\partial}{\partial x_k}\Delta_x \overline{u_i u_i u_k}$$
$$+ \frac{\nu}{2\rho}\frac{\partial}{\partial x_i}\Delta_x \overline{u_i p} + \frac{\nu^2}{2}\Delta_x \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} - \frac{\nu^2}{4}\Delta_x \Delta_x \overline{u_i u_i} = 0. \qquad \text{(A 16)}$$

Introducing the transformation

$$\frac{\nu^2}{2}\Delta_x \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} = \frac{\nu^2}{8}\Delta_x \Delta_x \overline{u_i u_i} - \frac{\nu^2}{2}\Delta_x (\Delta_\xi \overline{u_i u_i'})_0, \qquad \text{(A 17)}$$

(A16) reduces to the form

$$\frac{\nu}{4}\Delta_x \frac{\partial q^2}{\partial t} + \frac{\nu}{2}\frac{\partial U_k}{\partial x_l}\frac{\partial^2 q^2}{\partial x_l \partial x_k} + \frac{\nu}{4}U_k \frac{\partial}{\partial x_k}\Delta_x q^2 + \frac{\nu}{4}\frac{\partial q^2}{\partial x_k}\Delta_x U_k + \nu \frac{\partial \overline{u_i u_k}}{\partial x_l}\frac{\partial^2 U_i}{\partial x_k \partial x_l}$$
$$+ \frac{\nu}{2}(\Delta_x \overline{u_i u_k})\frac{\partial U_i}{\partial x_k} + \frac{\nu}{2}\overline{u_i u_k}\frac{\partial}{\partial x_k}\Delta_x U_i + \frac{\nu}{4}\frac{\partial}{\partial x_k}\Delta_x \overline{u_i u_i u_k}$$
$$+ \frac{\nu}{2\rho}\frac{\partial}{\partial x_i}\Delta_x \overline{u_i p} + \frac{\nu^2}{8}\Delta_x \Delta_x \overline{u_i u_i} - \frac{\nu^2}{2}\Delta_x (\Delta_\xi \overline{u_i u_i'})_0 - \frac{\nu^2}{4}\Delta_x \Delta_x \overline{u_i u_i} = 0. \qquad \text{(A 18)}$$

Subtracting (A18) from (A14), we obtain

$$- \nu \frac{\partial}{\partial t}(\Delta_\xi \overline{u_i u_i'})_0 - \nu U_k \frac{\partial}{\partial x_k}(\Delta_\xi \overline{u_i u_i'})_0 - \nu (\Delta_\xi \overline{u_k u_i'})_0 \frac{\partial U_i}{\partial x_k} - \nu (\Delta_\xi \overline{u_i u_k'})_0 \frac{\partial U_i}{\partial x_k}$$
$$- \frac{\nu}{2}\overline{u_i u_k}\Delta_x \frac{\partial U_i}{\partial x_k} - \frac{\nu}{4}\frac{\partial q^2}{\partial x_k}\Delta_x U_k + \nu \left(\frac{\partial}{\partial \xi_l}\overline{u_k u_i'}\right)_0 \frac{\partial^2 U_i}{\partial x_k \partial x_l} - \nu \left(\frac{\partial}{\partial \xi_l}\overline{u_i u_k'}\right)_0 \frac{\partial^2 U_i}{\partial x_l \partial x_k}$$
$$- 2\nu \left(\frac{\partial^2}{\partial \xi_l \partial \xi_k}\overline{u_i u_i'}\right)_0 \frac{\partial U_k}{\partial x_l} - \frac{\nu}{2}\frac{\partial}{\partial x_k}[(\Delta_\xi \overline{u_i u_k u_i'})_0 + (\Delta_\xi \overline{u_i u_k' u_i'})_0]$$
$$- \nu \left[\left(\frac{\partial}{\partial \xi_k}\Delta_\xi \overline{u_i u_i' u_k'}\right)_0 - \left(\frac{\partial}{\partial \xi_k}\Delta_\xi \overline{u_i u_k u_i'}\right)_0\right] - \frac{\nu}{2\rho}\frac{\partial}{\partial x_i}[(\Delta_\xi \overline{p u_i'})_0 + (\Delta_\xi \overline{u_i p'})_0]$$
$$- \frac{\nu}{\rho}\left[\left(\frac{\partial}{\partial \xi_i}\Delta_\xi \overline{u_i p'}\right)_0 - \left(\frac{\partial}{\partial \xi_i}\Delta_\xi \overline{p u_i'}\right)_0\right] + \frac{\nu^2}{2}\Delta_x (\Delta_\xi \overline{u_i u_i'})_0 + 2\nu^2 (\Delta_\xi \Delta_\xi \overline{u_i u_i'})_0 = 0. \qquad \text{(A 19)}$$

The deduced result is identical with (4.4), which defines the homogeneous part of the turbulent dissipation rate. It is obvious that there is no reason why we should keep the higher-order terms that appear in (A14). The treatment of the dissipation correlations utilizing the two-point correlation technique is an elegant way of eliminating the shortcomings mentioned above.

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